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Addendum and Corrigendum to "Euclidean Gibbs Measures of Interacting Quantum Anharmonic Oscillators"

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Abstract A generalization of Theorem 3.14 in [Euclidean Gibbs measures of interacting quantum anharmonic oscillators. J. Stat. Phys. 127:985–1047 (2007)] is given. It describes the uniqueness of Euclidean Gibbs measures of a system of quantum anharmonic oscillators in an external field. We also discuss in more detail the applicability of the Lee-Yang theorem in the theory of such systems, which includes a correction of Lemma 8.4.

Keywords Lee-Yang theorem · GHS inequality · Euclidean Gibbs measure

Our aim in this note is to prove a more general statement than Theorem 3.14 of [4] and to discuss the applicability of the Lee-Yang theorem to the model described by that statement. This in particular means a correction of Lemma 8.4 of [4]. In the sequel, by quoting statements or equations numbered like 3.14 or (2.1) we mean the corresponding items of [4]. All the notations we use here are either taken from [4] or explicitly defined here.

Uniqueness of Gibbs Measures In the new version of Theorem 3.14 presented below the statement is the same as in the 'old' one but the class of potentials is much larger.

Theorem 1 *Let the model be translation invariant and the anharmonic potential be of the form*

$$V(x) = V_0(x) - hx, \quad h \in \mathbb{R}, \tag{1}$$

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where V_0 is even, satisfies the stability condition (2.4), *and is such that the derivative $V'(x)$ is convex for $x \in (0, +\infty)$. Then the set $\mathcal{G}^{\mathrm{t}}$ is a singleton if $h \neq 0$.*

Proof Here instead of using arguments based on the Lee-Yang theorem, as we did in [4], we employ the GHS inequality. For the Ising model, this way was suggested long time ago by C. J. Preston in [6]. In spite of its simplicity and significance, it has been very rarely used since that time.

As in [4], we get the result by showing that the pressure $p(h)$ is differentiable at every $h \neq 0$, see Corollary 3.11. For $\Lambda \Subset \mathbb{L}$, we consider

$$M_\Lambda(h) = \frac{1}{|\Lambda|} \sum_{\ell \in \Lambda} \int_{\Omega_\Lambda} \left(\int_0^\beta \omega_\ell(\tau) \mathrm{d}\tau \right) \mu_\Lambda(\mathrm{d}\omega_\Lambda), \tag{2}$$

where μ_Λ is defined by (2.28)–(2.30) with $V_\ell = V$ as in (1). By (3.13) it follows that $p'_\Lambda(h) = M_\Lambda(h)$. For $\ell_1, \ldots, \ell_n \in \Lambda$ and $\tau_1, \ldots, \tau_n \in [0, \beta]$, we set

$$\begin{aligned} S^\Lambda_{\ell_1 \ldots \ell_n}(\tau_1, \ldots, \tau_n; h) &= \int_{\Omega_\Lambda} \omega_{\ell_1}(\tau_1) \cdots \omega_{\ell_n}(\tau_n) \mu_\Lambda(\mathrm{d}\omega_\Lambda) \\ &= \int_{\Omega} \omega_{\ell_1}(\tau_1) \cdots \omega_{\ell_n}(\tau_n) \pi_\Lambda(\mathrm{d}\omega|0), \end{aligned} \tag{3}$$

see (7.10) and (2.58). With the help of the Ursell function, c.f. (7.9),

$$\begin{aligned} U^\Lambda_{\ell_1\ell_2\ell_3}(\tau_1, \tau_2, \tau_3; h) = {} & S^\Lambda_{\ell_1\ell_2\ell_3}(\tau_1, \tau_2, \tau_3; h) - S^\Lambda_{\ell_1}(\tau_1; h) S^\Lambda_{\ell_2\ell_3}(\tau_2, \tau_3; h) \\ & - S^\Lambda_{\ell_2}(\tau_2; h) S^\Lambda_{\ell_1\ell_3}(\tau_1, \tau_3; h) - S^\Lambda_{\ell_3}(\tau_3; h) S^\Lambda_{\ell_1\ell_2}(\tau_1, \tau_2; h) \\ & + 2 S^\Lambda_{\ell_1}(\tau_1; h) S^\Lambda_{\ell_2}(\tau_2; h) S^\Lambda_{\ell_3}(\tau_3; h) \end{aligned} \tag{4}$$

one writes the derivative of (2) in the form

$$M''_\Lambda(h) = \frac{1}{|\Lambda|} \sum_{\ell_1, \ell_2, \ell_3 \in \Lambda} \int_0^\beta \int_0^\beta \int_0^\beta U^\Lambda_{\ell_1\ell_2\ell_3}(\tau_1, \tau_2, \tau_3; h) \mathrm{d}\tau_1 \mathrm{d}\tau_2 \mathrm{d}\tau_3. \tag{5}$$

For the potential (1), in the classical case the GHS inequality holds, see Theorem 12.12, p. 131 in [7] and also [3]. By the method of proving correlation inequalities for the Euclidean Gibbs measures discussed in [4], see also [2], this yields

$$U^\Lambda_{\ell_1\ell_2\ell_3}(\tau_1, \tau_2, \tau_3; h) \leq 0, \quad \text{for } h \geq 0.$$

Thus, $-M_\Lambda(h)$ is convex on $(0, +\infty)$. Let $\mathcal{L}$ be a van Hove sequence, see Definition 3.9. By (4.16), (4.14) and (2.36), (2.18) one obtains $|M_\Lambda(h)| \leq C$ for all $\Lambda \Subset \mathbb{L}$. The constant C may depend on h but can be estimated uniformly on the intervals $h \in (a, b)$, $a < b < +\infty$. Thereby, the sequence $\{M_\Lambda(h)\}_{\Lambda \in \mathcal{L}}$ is uniformly bounded on every (a, b). Then, for $a > 0$, by Lemma 1 of [6] it contains a subsequence such that $M_{\Lambda_n}(h) \to M(h)$ for all $h \in (a, b)$. By Theorem 3.10, for this subsequence, we have $p_{\Lambda_n}(h) \to p(h)$, which in view of the convexity of $p_\Lambda(h)$'s yields that $p(h)$ is differentiable on $h \in (a, b)$ and hence $|\mathcal{G}^{\mathrm{t}}| = 1$. As V_0 is even, this also yields the same result for $h < 0$. □

The Lee-Yang Theorem A more detailed consideration of the lattice approximation technique mentioned in the proof of Lemma 8.4 yields that one has to impose further restrictions on the potential $V(x) = v(x^2) - hx$. Thus, a statement employing the Lee-Yang property, see Definition 8.1, which we can prove is as follows.

Theorem 2 *Let the model be translation invariant and the anharmonic potential be such that the probability measure*

$$Z_b^{-1} \exp(-bx^2 - v(x^2))\mathrm{d}x, \quad x \in \mathbb{R},$$

has the Lee-Yang property for all $b \geq b_0$ with certain $b_0 \in \mathbb{R}$. Then the limiting pressure possesses holomorphic extensions to certain domains containing the real line, except possibly for the point $h = 0$.

The proof of this theorem is based on the lattice approximations of Euclidean Gibbs measures, see [2], in which $\exp(|\Lambda| p_\Lambda(h))$ is approximated by functions $f_N(h^2)$, $N \in \mathbb{N}$, each of which belongs to $\mathcal{F}_{\text{Laguerre}}$ (see (3.35)) and $f_N(h^2) \to \exp(|\Lambda| p_\Lambda(h))$ in such a way that the limiting pressure has the property stated. However, the condition of Lemma 8.4, that is $b + v' \in \mathcal{F}_{\text{Laguerre}}$ for certain $b \geq -a/2$, ensures that $f_N \in \mathcal{F}_{\text{Laguerre}}$ but for finitely many N's only. This in particular means the validity of Lemma 8.4, and hence of Theorem 3.14 as it is given in [4], for the corresponding classical model, which one obtains by letting $m \to +\infty$, see [1].

It turns out that the class of functions possessing the property as in Theorem 2 can be described explicitly. By C. M. Newman, see Theorem 2 in [5], we have that v obeys the condition of Theorem 2 if and only if it has the form

$$v(x^2) = \alpha x^4 + \beta x^2 - \log \Phi(x^2),$$

where

$$\Phi(t) = \prod_{k \in K} \phi(t\gamma_k), \qquad \phi(t) = (1 + t)e^{-t},$$

The set K may be void, finite, or countable; the numbers $\gamma_k > 0$ obey the condition $\sum_{k \in K} \gamma_k^2 < \infty$; $\alpha \geq 0$. If $\alpha > 0$, the number β may be arbitrary real; if $\alpha = 0$, one demands $\beta + \sum_{k \in K} \gamma_k > 0$. Thus, the only possibility for v to have the property $b + v' \in \mathcal{F}_{\text{Laguerre}}$ for all $b \geq b_0$ is $v(x^2) = \alpha x^4 + \beta x^2$, $\alpha > 0$, $\beta \in \mathbb{R}$, which is the well-known case ϕ^4.

References

1. Albeverio, S., Kondratiev, Y.G., Kozitsky, Y.: Classical limits of Euclidean Gibbs states for quantum lattice models. Lett. Math. Phys. **48**, 221–233 (1999)
2. Albeverio, S., Kondratiev, Y.G., Kozitsky, Y., Röckner, M.: Euclidean Gibbs states of quantum lattice systems. Rev. Math. Phys. **14**, 1–67 (2002)
3. Ellis, R.S., Monroe, J.L., Newman, C.M.: The GHS and other correlation inequalities for a class of even ferromagnets. Commun. Math. Phys. **46**, 167–182 (1976)
4. Kozitsky, Y., Pasurek, T.: Euclidean Gibbs measures of interacting quantum anharmonic oscillators. J. Stat. Phys. **127**, 985–1047 (2007)
5. Newman, C.M.: Fourier transforms with only real zeros. Proc. Am. Math. Soc. **61**, 245–251 (1976)
6. Preston, C.J.: An application of the GHS inequalities to show the absence of phase transition for Ising spin systems. Commun. Math. Phys. **35**, 253–255 (1974)
7. Simon, B.: Functional Integration and Quantum Physics. Academic Press, New York (1979)